

BIHARMONIC HYPERSURFACES IN SPACE FORMS WITH THREE DISTINCT PRINCIPAL CURVATURES

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ABSTRACT. In this paper, we have studied biharmonic hypersurfaces in space form $\overline{M}^{n+1}(c)$ with constant sectional curvature c . We have obtained that biharmonic hypersurfaces M^n with at most three distinct principal curvatures in $\overline{M}^{n+1}(c)$ has constant mean curvature. We also obtain the full classification of biharmonic hypersurfaces with at most three distinct principal curvatures in arbitrary dimension space form $\overline{M}^{n+1}(c)$.

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1. Introduction

The longstanding well known Chen's conjecture on biharmonic submanifolds states that a biharmonic submanifold in a Euclidean space is a minimal one [2]. In particular, Chen proved that there exist no proper biharmonic surfaces in Euclidean 3-spaces. There are many non-existence results in Euclidean spaces developed by I. Dimitric in [9, 10]. Later, the Chen's conjecture was verified and found true for submanifolds of some Euclidean spaces (see [7, 12, 13, 14]).

In contrast to the submanifolds of Euclidean spaces, Chen's conjecture is not always true for the submanifolds of the pseudo-Euclidean spaces (see [3~6]). However, for hypersurfaces in pseudo-Euclidean spaces, Chen's conjecture is also right (see [1, 8]).

For biharmonic hypersurfaces with at most two distinct principal curvatures the property of having constant mean curvature was proved in [15] for any space form. This property proved to be the main ingredient for the following complete classification of proper biharmonic hypersurfaces with at most two distinct principal curvatures in the Euclidean sphere.

Theorem 1.1 ([15]): Let M^m be a proper biharmonic hypersurface with at most two distinct principal curvatures in \mathbb{S}^{m+1} . Then M is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ or of $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

Proposition 1.2 ([15]): Let M^m be a proper biharmonic hypersurface with constant mean curvature H in \mathbb{S}^{m+1} . Then M has constant scalar curvature,

$$s = m^2(1 + k) - 2m,$$

where $H^2 = k$.

For biharmonic hypersurfaces in 4-dimensional space form the property of having constant mean curvature was proved in [19] and the following classification result was obtained

Theorem 1.3 ([19]): There exist no compact proper biharmonic hypersurfaces of

constant mean curvature and with three distinct principal curvatures in the unit Euclidean sphere.

Theorem 1.4([19]): The only compact proper biharmonic hypersurfaces of \mathbb{S}^4 are the hypersphere $\mathbb{S}^3(\frac{1}{\sqrt{2}})$ and the torus $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^2(\frac{1}{\sqrt{2}})$.

In view of above development, we study the biharmonic hypersurfaces in $\overline{M}^{n+1}(c)$ with at most three distinct principal curvatures.

2. Preliminaries

Let (M^n, g) be a hypersurface isometrically immersed in a $(n+1)$ -dimensional space forms $(\overline{M}^{n+1}(c), \overline{g})$ with constant sectional curvature c and $g = \overline{g}|_M$.

Let $\overline{\nabla}$ and ∇ denote linear connections on $\overline{M}^{n+1}(c)$ and M^n , respectively. Then, the Gauss and Weingarten formulae are given by

$$(2.1) \quad \overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

$$(2.2) \quad \overline{\nabla}_X \xi = -A_\xi X,$$

where ξ be the unit normal vector to M , h is the second fundamental form and A is the shape operator. It is well known that the second fundamental form h and shape operator A are related by

$$(2.3) \quad \overline{g}(h(X, Y), \xi) = g(A_\xi X, Y).$$

The mean curvature vector is given by

$$(2.4) \quad H = \frac{1}{n} \text{trace} A.$$

The Gauss and Codazzi equations are given by

$$(2.5) \quad R(X, Y)Z = c(g(Y, Z)X - g(X, Z)Y) + g(AY, Z)AX - g(AX, Z)AY,$$

$$(2.6) \quad (\nabla_X A)Y = (\nabla_Y A)X,$$

respectively, where R is the curvature tensor and

$$(2.7) \quad (\nabla_X A)Y = \nabla_X AY - A(\nabla_X Y)$$

for all $X, Y, Z \in \Gamma(TM)$.

A biharmonic submanifold in a space form $\overline{M}(c)$ is called proper biharmonic if it is not minimal. The necessary and sufficient conditions for M to be proper biharmonic in $\overline{M}^{n+1}(c)$ [3, 17] is

$$(2.8) \quad \Delta H - H(nc - \text{trace} A^2) = 0,$$

$$(2.9) \quad 2A \text{grad} H + nH \text{grad} H = 0,$$

where H denotes the mean curvature. Also the Laplace operator Δ of a scalar valued function f is given by [4]

$$(2.10) \quad \Delta f = - \sum_{i=1}^n (e_i e_i f - \nabla_{e_i} e_i f),$$

where $\{e_1, e_2, \dots, e_n\}$ is an orthonormal local tangent frame on M^n .

We recall that a hypersurface M^n in \mathbb{S}^{n+1} is said to be isoparametric of type l if it has constant principal curvatures $k_1 > \dots > k_l$ with respective constant multiplicities n_1, \dots, n_l , $n = n_1 + n_2 + \dots + n_l$. It is known that the number l is either 1, 2, 3, 4 or 6. For $l \leq 3$, we have the following classification of compact isoparametric hypersurfaces. If $l = 1$, then M is totally umbilical. If $l = 2$, then $M = \mathbb{S}^{n_1}(r_1) \times \mathbb{S}^{n_2}(r_2)$, $r_1^2 + r_2^2 = 1$ (see [18]). If $l = 3$, then $n_1 = n_2 = n_3 = 2^q$, $q = 0, 1, 2, 3$ (see [16]).

Moreover, there exists an angle θ , $0 < \theta < \frac{\pi}{l}$, such that

$$(2.11) \quad k_\alpha = \cot\left(\theta + \frac{(\alpha-1)\pi}{l}\right), \quad \alpha = 1, \dots, l.$$

In the next section, we shall need the following result:

Theorem 2.1 ([11]): A compact hypersurface M^m of constant scalar curvature s and constant mean curvature H in \mathbb{S}^{m+1} is isoparametric provided it has 3 distinct principal curvatures everywhere.

3. Biharmonic hypersurfaces with three distinct principal curvatures

In this section, we study biharmonic hypersurfaces M in space form $\overline{M}^{n+1}(c)$. We assume that H is not constant. The hypothesis for M to be proper biharmonic with three distinct principal curvatures in space form $\overline{M}^{n+1}(c)$ and non-constant mean curvature, implies the existence of an open connected subset U of M , with $\text{grad}_p H \neq 0$ for all $p \in U$. We shall contradict the condition $\text{grad}_p H \neq 0$, $\forall p \in U$. From (2.9), it is easy to see that $\text{grad}H$ is an eigenvector of the shape operator A with the corresponding principal curvature $\frac{-nH}{2}$. We choose e_1 in the direction of $\text{grad}H$ and therefore shape operator A of hypersurfaces will take the following form with respect to a suitable frame $\{e_1, e_2, \dots, e_{n-1}, e_n\}$

$$(3.1) \quad A_H = \begin{pmatrix} \frac{-nH}{2} & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \lambda_{n-1} & \\ & & & & & \lambda_n \end{pmatrix}.$$

The $\text{grad}H$ can be expressed as

$$(3.2) \quad \text{grad}H = \sum_{i=1}^n e_i(H)e_i.$$

As we have taken e_1 parallel to $\text{grad}H$, consequently

$$(3.3) \quad e_1(H) \neq 0, e_2(H) = 0, e_3(H) = 0, \dots, e_{n-1}(H) = 0, e_n(H) = 0.$$

We express

$$(3.4) \quad \nabla_{e_i} e_j = \sum_{k=1}^n \omega_{ij}^k e_k, \quad i, j = 1, 2, \dots, n.$$

Using (3.4) and the compatibility conditions $(\nabla_{e_k} g)(e_i, e_i) = 0$ and $(\nabla_{e_k} g)(e_i, e_j) = 0$, we obtain

$$(3.5) \quad \omega_{ki}^i = 0, \quad \omega_{ki}^j + \omega_{kj}^i = 0,$$

for $i \neq j$, and $i, j, k = 1, 2, \dots, n$.

Taking $X = e_i, Y = e_j$ in (2.7) and using (3.1), (3.4), we get

$$(\nabla_{e_i} A)e_j = e_i(\lambda_j)e_j + \sum_{k=1}^n \omega_{ij}^k e_k(\lambda_j - \lambda_k).$$

Putting the value of $(\nabla_{e_i} A)e_j$ in (2.6), we find

$$e_i(\lambda_j)e_j + \sum_{k=1}^n \omega_{ij}^k e_k(\lambda_j - \lambda_k) = e_j(\lambda_i)e_i + \sum_{k=1}^n \omega_{ji}^k e_k(\lambda_i - \lambda_k),$$

whereby for $i \neq j = k$ and $i \neq j \neq k$, we obtain

$$(3.6) \quad e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j,$$

$$(3.7) \quad (\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j,$$

respectively, for distinct $i, j, k = 1, 2, \dots, n$.

Since $\lambda_1 = \frac{-nH}{2}$, from (3.3), we get

$$(3.8) \quad e_1(\lambda_1) \neq 0, e_2(\lambda_1) = 0, e_3(\lambda_1) = 0, \dots, e_{n-1}(\lambda_1) = 0, e_n(\lambda_1) = 0.$$

Using (3.8), we have

$$[e_i, e_j](\lambda_1) = 0, \quad i, j = 2, \dots, n,$$

whereby using (3.4), we find

$$(3.9) \quad \omega_{ij}^1 = \omega_{ji}^1,$$

for $i \neq j$ and $i, j = 2, \dots, n$.

Now we show that $\lambda_j \neq \lambda_1, j = 2, 3, \dots, n$. In fact, if $\lambda_j = \lambda_1$ for $j \neq 1$, from (3.6), we find

$$(3.10) \quad e_1(\lambda_j) = (\lambda_1 - \lambda_j)\omega_{j1}^j = 0,$$

which contradicts the first expression of (3.8).

Since M^n has three distinct principal curvatures, we can assume that $\lambda_2 = \lambda_3 = \dots = \lambda_{n-1} = \lambda \neq \lambda_n$. From (2.4), we obtain that

$$(3.11) \quad \lambda_n = \frac{3nH}{2} - (n-2)\lambda, \quad \lambda \neq \frac{-nH}{2}, \frac{2nH}{n-2}, \frac{3nH}{2(n-1)}.$$

Putting $i, j = 2, 3, \dots, n-1$, and $i \neq j$ in (3.6), we get

$$(3.12) \quad e_j(\lambda) = 0, \quad \text{for } j = 2, 3, \dots, n-1.$$

Putting $i \neq 1, j = 1$ in (3.6) and using (3.8) and (3.5), we find

$$(3.13) \quad \omega_{1i}^1 = 0, \quad i = 1, 2, 3, \dots, n.$$

Putting $i = 2, 3, \dots, n-1, j = n$ in (3.6) and using (3.12), we obtain

$$(3.14) \quad \omega_{ni}^n = 0, \quad i = 2, 3, \dots, n-1.$$

Putting $i = 1, j = 2, 3, \dots, n-1, n$, in (3.6), we have

$$(3.15) \quad \omega_{n1}^n = \frac{e_1(3nH - 2(n-2)\lambda)}{-4nH + 2(n-2)\lambda}, \quad \omega_{j1}^j = -\frac{2e_1(\lambda)}{nH + 2\lambda}, \quad j = 2, 3, \dots, n-1.$$

Putting $i = n, j = 2, 3, \dots, n-1$, in (3.6), we find

$$(3.16) \quad \omega_{jn}^j = \frac{2e_n(\lambda)}{3nH - 2(n-1)\lambda}, \quad j = 2, 3, \dots, n-1.$$

Putting $i = 1, j \neq k$, and $j, k = 2, 3, \dots, n-1$, in (3.7), we obtain

$$(3.17) \quad \omega_{k1}^j = 0, \quad j \neq k, \quad \text{and} \quad j, k = 2, 3, \dots, n-1.$$

Putting $i = n, j \neq k$, and $j, k = 2, 3, \dots, n-1$, in (3.7), we have

$$(3.18) \quad \omega_{kn}^j = 0, \quad j \neq k, \quad \text{and} \quad j, k = 2, 3, \dots, n-1.$$

Putting $i = n, j = 1$, and $k = 2, 3, \dots, n-1$, in (3.7), and using (3.9) we get

$$(3.19) \quad \omega_{kn}^1 = \omega_{nk}^1 = 0, \quad k = 2, 3, \dots, n-1.$$

Putting $i = 1, j = n$, and $k = 2, 3, \dots, n-1$, in (3.7), and using (3.9) we find

$$(3.20) \quad \omega_{1k}^n = \omega_{k1}^n = 0, \quad k = 2, 3, \dots, n-1.$$

Now, we have the following:

Lemma 3.1. *Let M^n be an n -dimensional biharmonic hypersurface with three distinct principal curvatures and non-constant mean curvature in space forms $\overline{M}^{n+1}(c)$, having the shape operator given by (3.1) with respect to suitable orthonormal frame $\{e_1, e_2, \dots, e_{n-1}, e_n\}$. Then, we obtain*

$$(3.21) \quad \nabla_{e_1} e_1 = 0, \quad \nabla_{e_i} e_1 = -\alpha e_i, \quad i = 2, 3, \dots, n-1, \quad \nabla_{e_n} e_1 = \beta e_n,$$

$$(3.22) \quad \nabla_{e_i} e_i = \alpha e_1 + \sum_{i \neq j, j=2}^{n-1} \omega_{ii}^j e_j - \frac{2e_n(\lambda)}{3nH - 2(n-1)\lambda} e_n, \quad i = 2, 3, \dots, n-1,$$

$$(3.23) \quad \nabla_{e_i} e_j = \sum_{i \neq j, k=2}^{n-2} \omega_{ij}^k e_k, \quad i, j = 2, 3, \dots, n-1,$$

$$(3.24) \quad \nabla_{e_1} e_n = 0, \quad \nabla_{e_n} e_n = -\beta e_1, \quad \nabla_{e_i} e_n = \frac{2e_n(\lambda)}{3nH - 2(n-1)\lambda} e_i, \quad i = 2, 3, \dots, n-1,$$

where ω_{ij}^k satisfies (3.5) for $i, j, k = 1, 2, 3, \dots, n-1, n$, and $\alpha = \frac{2e_1(\lambda)}{nH + 2\lambda}$, $\beta = \frac{e_1(3nH - 2(n-2)\lambda)}{-4nH + 2(n-2)\lambda}$.

Using Lemma 3.1, Gauss equation and comparing the coefficients with respect to an orthonormal frame $\{e_1, e_2, \dots, e_{n-1}, e_n\}$, we find the following:

$$\bullet X = e_1, Y = e_2, Z = e_1,$$

$$(3.25) \quad e_1(\alpha) = \alpha^2 + c - \frac{nH\lambda}{2}.$$

$$\bullet X = e_1, Y = e_2, Z = e_n,$$

$$(3.26) \quad e_1\left(\frac{2e_n(\lambda)}{3nH - 2(n-1)\lambda}\right) - \alpha \frac{2e_n(\lambda)}{3nH - 2(n-1)\lambda} = 0.$$

$$\bullet X = e_1, Y = e_n, Z = e_1,$$

$$(3.27) \quad e_1(\beta) = -\beta^2 - c + \frac{nH}{2} \left(\frac{3nH}{2} - (n-2)\lambda \right).$$

$$\bullet X = e_3, Y = e_n, Z = e_1,$$

$$(3.28) \quad e_n(\alpha) + \frac{2e_n(\lambda)}{3nH - 2(n-1)\lambda}(\alpha + \beta) = 0.$$

$$\bullet X = e_n, Y = e_2, Z = e_n,$$

$$(3.29) \quad e_n \left(\frac{2e_n(\lambda)}{3nH - 2(n-1)\lambda} \right) - \alpha\beta - \left(\frac{2e_n(\lambda)}{3nH - 2(n-1)\lambda} \right)^2 = -c - \lambda \left(\frac{3nH}{2} - (n-2)\lambda \right).$$

Using (2.8), (2.10), (3.1) and Lemma 3.1, we find

$$(3.30) \quad -e_1e_1(H) + [(n-2)\alpha - \beta]e_1(H) + H \left[\frac{n^2H^2}{4} + (n-2)\lambda^2 + \left(\frac{3nH}{2} - (n-2)\lambda \right)^2 \right] - ncH = 0.$$

From (3.3) and Lemma 3.1, we obtain

$$(3.31) \quad e_i e_1(H) = 0, \quad i = 2, 3, \dots, n-1, n.$$

Differentiating $\alpha = \frac{2e_1(\lambda)}{nH+2\lambda}$, $\beta = \frac{e_1(3nH-2(n-2)\lambda)}{-4nH+2(n-2)\lambda}$ along e_n , we get equations

$$(nH + 2\lambda)e_n(\alpha) + 2\alpha e_n(\lambda) = 2e_n e_1(\lambda),$$

$$(-4nH + 2(n-2)\lambda)e_n(\beta) = -2(n-2)e_n e_1(\lambda) - 2(n-2)\beta e_n(\lambda)$$

respectively and eliminating $e_n e_1(\lambda)$, we have

$$(-4nH + 2(n-2)\lambda)e_n(\beta) = -(n-2)(nH + 2\lambda)e_n(\alpha) - 2(n-2)(\alpha + \beta)e_n(\lambda).$$

Putting the value of $e_n(\alpha)$ from (3.28) in the above equation, we find

$$e_n(\beta) = \frac{4e_n(\lambda)n(n-2)(\alpha+\beta)(\lambda-H)}{(-4nH+2(n-2)\lambda)(3nH-(2n-2)\lambda)}.$$

Differentiating (3.30) along e_n and using (3.31), (3.28) and $e_n(\beta)$, we get

$$(3.32) \quad e_n(\lambda) \left[\frac{4(\alpha + \beta)e_1(H)}{-4nH + 2(n-2)\lambda} + H((2n-2)\lambda - 3nH) \right] = 0.$$

We claim that $e_n(\lambda) = 0$. Indeed, if $e_n(\lambda) \neq 0$, then

$$(3.33) \quad \frac{4(\alpha + \beta)e_1(H)}{-4nH + 2(n-2)\lambda} + H((2n-2)\lambda - 3nH) = 0.$$

Now, differentiating (3.33) along e_n , we have

$$(3.34) \quad \frac{8(\alpha + \beta)(nH(14-5n) + 4(n-2)(n-1)\lambda)e_1(H)}{(-4nH + 2(n-2)\lambda)^2 \left(\frac{3nH}{2} - (n-2)\lambda \right)} + H((2n-2)\lambda - 3nH) = 0.$$

Eliminating $e_1(H)$ from (3.33) and (3.34), we obtain

$$2(n-1)\lambda - 3nH = 0$$

which is not possible since $\lambda \neq \frac{3nH}{2(n-1)}$, consequently, $e_n(\lambda) = 0$. Therefore, (3.29) reduces to

$$(3.35) \quad \alpha\beta = c + \lambda \left(\frac{3nH}{2} - (n-2)\lambda \right).$$

Now, eliminating $e_1 e_1(H)$ and $e_1 e_1(\lambda)$, using (3.35), (3.30), (3.27) and (3.25), we obtain

$$(3.36) \quad [(10n-2n^2)\alpha-4n\beta]e_1(H) = \frac{21n^3H^3}{2} + 6(n^3-2n^2)H\lambda^2 + (-15n^3+18n^2)H^2\lambda - 6(n^2+n)cH.$$

Differentiating (3.36) along e_1 and using (3.35), (3.30), (3.27), (3.25) and (3.36), we get

$$(3.37) \quad \begin{aligned} & [(13n^3 + \frac{11n^2}{2})H^3 + (4n^3 - 14n^2 + 2n + 20)H\lambda^2 + (-15n^3 + 18n^2 + 24n)H^2\lambda + cH(2n^3 \\ & \quad - 16n^2 - 6n)]\alpha + [-31n^2H^3 + (-16n^2 + 36n - 8)H\lambda^2 + (42n^2 - 60n)H^2\lambda \\ & \quad + cH(10n^2 + 6n)]\beta = e_1(H)[\frac{69n^2H^2}{2} + (24n - 30n^2)H\lambda + (6n + 4n^2 - 28)\lambda^2 - c(4n^2 + 20n)]. \end{aligned}$$

Also, we have

$$(3.38) \quad 3ne_1(H) = \alpha(n-2)(nH+2\lambda) + \beta(-4nH+2(n-2)\lambda)$$

Combining (3.37) and (3.38), we obtain

$$(3.39) \quad \begin{aligned} & [(9n^3 + 171n^2)H^3 + (16n^3 + 40n^2 - 244n - 200)H\lambda^2 + (-30n^3 - 198n^2 - 516n)H^2\lambda \\ & \quad - (16n^2 - 8n - 160 + \frac{224}{n})\lambda^3 + cH(20n^3 - 72n^2 - 116n) + c\lambda(16n^2 + 48n - 160)]\alpha \\ & \quad + [90n^2H^3 + (56n^2 - 72n - 80)H\lambda^2 + (-126n^2 + 108n)H^2\lambda - (16n^2 - 8n - 160 + \frac{224}{n})\lambda^3 \\ & \quad + cH(28n^2 - 124n) + c\lambda(16n^2 + 48n - 160)]\beta = 0. \end{aligned}$$

For simplicity, we denote by

$$\begin{aligned} p_1 &= (9n^3 + 171n^2)H^3 + (16n^3 + 40n^2 - 244n - 200)H\lambda^2 + (-30n^3 - 198n^2 - 516n)H^2\lambda - (16n^2 - 8n - 160 + \frac{224}{n})\lambda^3 + cH(20n^3 - 72n^2 - 116n) + c\lambda(16n^2 + 48n - 160) \\ p_2 &= 90n^2H^3 + (56n^2 - 72n - 80)H\lambda^2 + (-126n^2 + 108n)H^2\lambda - (16n^2 - 8n - 160 + \frac{224}{n})\lambda^3 + cH(28n^2 - 124n) + c\lambda(16n^2 + 48n - 160). \end{aligned}$$

Therefore, (3.39) can be rewritten as

$$(3.40) \quad \alpha p_1 + \beta p_2 = 0.$$

On the other hand, combining (3.38) with (3.36) and using (3.35), we find

$$(3.41) \quad \alpha^2(n-2)(10-2n)(nH+2\lambda) - 4\beta^2(-4nH+2(n-2)\lambda) = L,$$

where L is given by

$$L = \frac{63n^3H^3}{2} + (28n^3 - 106n^2 + 100n)H\lambda^2 + (102n^2 - 51n^3)H^2\lambda - (4n^3 - 28n^2 + 64n - 48)\lambda^3 + cH(14n - 22n^2) + c\lambda(4n^2 - 20n + 24).$$

Using (3.40) and (3.35), we get

$$\alpha^2 = -\frac{p_2}{p_1}(c + \lambda(\frac{3nH}{2} - (n-2)\lambda)), \quad \beta^2 = -\frac{p_1}{p_2}(c + \lambda(\frac{3nH}{2} - (n-2)\lambda))$$

Eliminating α^2 and β^2 from (3.41), we obtain

$$(3.42) \quad (c + \frac{3nH\lambda}{2} - (n-2)\lambda^2)[(14n-2n^2-20)(nH+2\lambda)p_2^2 - 4p_1^2(-4nH+2(n-2)\lambda)] = Lp_1p_2,$$

which is a polynomial equation of degree 9 in terms of λ and H .

Now consider an integral curve of e_1 passing through $p = \gamma(t_0)$ as $\gamma(t)$, $t \in I$. Since $e_i(H) = e_i(\lambda) = 0$ for $i = 2, \dots, n$ and $e_1(H), e_1(\lambda) \neq 0$, we can assume $t = t(\lambda)$

and $H = H(\lambda)$ in some neighborhood of $\lambda_0 = \lambda(t_0)$. Using (3.38) and (3.40), we have

$$\begin{aligned}
 (3.43) \quad \frac{dH}{d\lambda} &= \frac{dH}{dt} \frac{dt}{d\lambda} = \frac{e_1(H)}{e_1(\lambda)} \\
 &= \frac{2(\alpha(n-2)(nH+2\lambda)+\beta(-4nH+2(n-2)\lambda))}{3n\alpha(nH+2\lambda)} \\
 &= \frac{2(n-2)}{3n} + \frac{p_1(4nH-2(n-2)\lambda)}{3n(nH+2\lambda)p_2}
 \end{aligned}$$

Differentiating (3.42) with respect to λ and substituting $\frac{dH}{d\lambda}$ from (3.43), we get

$$(3.44) \quad f(H, \lambda) = 0,$$

another algebraic equation of degree 12 in terms of H and λ . We rewrite (3.42) and (3.44) respectively in the following forms

$$(3.45) \quad \sum_{i=0}^9 f_i(H)\lambda^i = 0, \quad \sum_{j=0}^{12} g_j(H)\lambda^j = 0,$$

where $f_i(H)$ and $g_j(H)$ are polynomial functions of H . We eliminate λ^{12} between these two polynomials of (3.45) by multiplying $g_{12}\lambda^3$ and f_8 respectively on the first and second equations of (3.45), we obtain a new polynomial equation in λ of degree 11. Combining this equation with the first equation of (3.45), we successively obtain a polynomial equation in λ of degree 10. In a similar way, by using the first equation of (3.45) and its consequences we are able to gradually eliminate λ . At last, we obtain a non-trivial algebraic polynomial equation in H with constant coefficients. Therefore, we conclude that the real function H must be a constant and we conclude:

Theorem 3.2. *Every biharmonic hypersurface M in the space forms $\overline{M}^{n+1}(c)$ with three distinct principal curvatures must be of constant mean curvature.*

Combining Theorem 3.2 with Theorem 4.1 [15], we obtain that

Theorem 3.3. *Every biharmonic hypersurface M in the space forms $\overline{M}^{n+1}(c)$ with at most three distinct principal curvatures must be of constant mean curvature.*

Theorem 3.4. *There exist no proper biharmonic hypersurfaces M with at most three distinct principal curvatures in \mathbb{H}^{n+1} or \mathbb{R}^{n+1} .*

Proof: Suppose that M^n is a proper biharmonic hypersurface in \mathbb{H}^{n+1} or \mathbb{R}^{n+1} with at most three distinct principal curvatures. From Theorem 3.3, we have that mean curvature of M^n is constant. From (2.8), we get that $\text{trace}A^2 = -n$ or $\text{trace}A^2 = 0$, which is not possible and proof of the theorem is complete.

Theorem 3.5. *The only compact proper biharmonic hypersurfaces with at most three distinct principal curvatures of $\mathbb{S}^{n+1}(1)$ are the hypersphere $\mathbb{S}^n(\frac{1}{\sqrt{2}})$ and the torus $\mathbb{S}^{n_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{n_2}(\frac{1}{\sqrt{2}})$ where $n_1 + n_2 = n$, $n_1 \neq n_2$.*

Proof: Suppose that M^n is a compact proper biharmonic hypersurface of $\mathbb{S}^{n+1}(1)$ with three distinct principal curvatures. From Theorem 3.2, we get that M^n has constant mean curvature and, since it satisfies the hypotheses of Proposition 1.2, we conclude that it also has constant scalar curvature. We can thus apply Theorem 2.1 and it results that M^n is isoparametric in $\mathbb{S}^{n+1}(1)$. From Theorem 1.3, we get that M^n cannot be isoparametric with $l = 3$, and by using Theorem 1.1 we conclude the proof.

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